Mean Square Projection Error Gradient-based Variable Forgetting Factor FAPI

Young-Kwang Seo, Jong-Woo Shin, Wongi Seo, and Hyoung-Nam Kim

***Abstract*— This paper proposes a fast subspace tracking method named gradient-based variable forgetting factor fast approximated power iteration (GVFF FAPI). Since the conventional FAPI uses a constant forgetting factor for estimating covariance matrix of source signals, it has a difficulty in being applied to non-stationary environments such as continuously time varying subspace. To cope with the drawback of the conventional FAPI method, we modified the forgetting factor control equation of the GVFF recursive least squares algorithm and then applied to the FAPI method. Simulation results show that the proposed GVFF FAPI algorithm is superior to the conventional FAPI in terms of subspace error and root mean square error (RMSE) of tracked direction of arrival.**

***Keywords*—** FAPI, GVFF RLS, subspace tracking, PAST

I. Introduction

Subspace tracking methods are an important pre-processing step to reduce the computational complexity in the field of subspace-based adaptive systems [1]. Instead of updating whole eigen-structure, subspace-tracking methods work only with the signal or noise subspace. This makes the subspace-tracking methods more efficient than conventional subspace estimation methods such as eigenvalue decomposition (EVD) or singular value decomposition (SVD) [1].

A review of historical advances in the subspace tracking methods can be found in [2]. In early 1990s, it turned out that the signal subspace tracking problem can be solved with a dominant complexity of only $O(nr)$ computations per each update. These kinds of methods are called by a fast subspace tracking method and a projection approximation subspace tracker (PAST) is the most popular one [3]. Although the PAST outperforms other methods in terms of tracking accuracy and global convergence property, it still suffers from the lack of orthonormality in estimated subspace columns. In order to reduce the orthonormality error of the PAST, a number of refined methods including OPAST (Orthonormal PAST) [4], NIC (Novel Information Criterion) [5], and NP3 (Natural Power method 3) [6] have been presented.

The final state of the development along this baseline has been recently marked by an advent of the so-called fast approximated power iteration (FAPI) [7]. The FAPI shows excellent performance in global convergence property, tracking accuracy, and orthonormality error even when the subspace of input data suddenly changes with time. However, since the FAPI involves a constant forgetting factor (FF) when estimating a covariance matrix as used in the conventional recursive least squares (RLS) algorithm, it could not cope well with continuously time-varying environments. The convergence rate of the FAPI is slow when the FF is close to 1, whereas the misadjustment is large when the FF is small. Therefore, the FAPI cannot simultaneously achieve both the fast convergence rate and the small misadjustment due to the constant FF.

In order to enhance the convergence performance of the FAPI method, this paper applies the gradient-based variable forgetting factor (GVFF) RLS algorithm to the FAPI method [8]. The GVFF RLS decreases the FF value when there is a large increase in mean square projection error (MSPE) and forces the FF toward the ceiling when there is negative gradient of MSPE. In addition, in order to improve the tracking performance in continuously time-varying environments, we modify the FF control equation in the manner that the FF has rapidly decreased for a positive gradient and slowly increased for negative gradient.

This paper is organized as follows. In Section II, the subspace tracking methods, the PAST, and the FAPI, are briefly described. Section III introduces the GVFF RLS algorithm and then gives a proposed subspace tracking method called the GVFF FAPI. Section IV presents some simulation results to demonstrate the performance of GVFF FAPI. Finally, the conclusions of this paper are summarized in Section V.

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II. Overview of Subspace Tracking methods

A. Projection Approximation Subspace Tracker (PAST)[3]

The goal of the PAST algorithm is to recursively estimate the principal subspace which is spanned by the eigenvectors associated with r dominant eigenvalues of a time-recursively updated covariance matrix $\mathbf{C}_{\mathbf{xx}}(t)$ of dimension $N \times N$ as follows:

$$\mathbf{C_{xx}}(t) = \beta \mathbf{C_{xx}}(t-1) + \mathbf{x}(t)\mathbf{x}^H(t)\,, \quad (1)$$

where $\mathbf{x}(t)$ is a data snapshot vector of dimension $n > r$ and β is a positive exponential forgetting factor close to one. An estimated subspace matrix $\mathbf{W}(t)$ is obtained as solving the minimization problem of exponentially weighted cost function defined by

$$J(\mathbf{W}(t)) = \sum_{u}^{t} \beta^{t-u} \left\| \mathbf{x}(u) - \mathbf{W}(t)\mathbf{W}^H(t)\mathbf{x}(u) \right\|^2 . \quad (2)$$

All sample vectors available in the time interval $1 \le u \le t$ are involved in estimating the signal subspace at the time instant *t*. The key issue of the PAST is to approximate $\mathbf{W}^H(t)\mathbf{x}(u)$ in (2). In other words, an input data vector $\mathbf{x}(u)$ is projected onto the column space of $\mathbf{W}(t)$ by the expression of $\mathbf{y}(u) = \mathbf{W}^H(u-1)\mathbf{x}(u)$. This results in a modified cost function defined by

$$J'(\mathbf{W}(t)) = \sum_{u}^{t} \beta^{t-u} \left\| \mathbf{x}(u) - \mathbf{W}(t)\mathbf{y}(u) \right\|^2 . \quad (3)$$

The computation of $\mathbf{W}(t)$ consists of a data compression step (4) and an orthonormalization step (5) of a compressed matrix at each update as

$$\mathbf{C_{xy}}(t) = \mathbf{C_{xx}}(t)\mathbf{W}(t-1)\,, \quad (4)$$
$$\mathbf{W}(t)\mathbf{R}(t) = \mathbf{C_{xy}}(t)\,, \quad (5)$$

where $\mathbf{C_{xy}}(t)$ is an $n \times r$ correlation matrix of an n dimensional data vector $\mathbf{x}(t)$ and a r dimensional compressed data vector $\mathbf{y}(t)$ as

$$\mathbf{y}(t) = \mathbf{W}(t-1)\mathbf{x}(t)\,. \quad (6)$$

The exponentially weighted least squares problem in (3) is well studied in field of adaptive signal processing. The modified cost function $J'(\mathbf{W}(t))$ is minimized if $\mathbf{R}(t)$ is equal to $\mathbf{C_{yy}}(t)$ as follows:

$$\mathbf{W}(t) = \mathbf{C_{xy}}(t)\mathbf{C_{yy}^{-1}}(t)\,, \quad (7)$$
$$\mathbf{C_{xy}}(t) = \beta \mathbf{C_{xy}}(t-1) + \mathbf{x}(t)\mathbf{y}^H(t)\,, \quad (8)$$
$$\mathbf{C_{yy}}(t) = \beta \mathbf{C_{yy}}(t-1) + \mathbf{y}(t)\mathbf{y}^H(t)\,. \quad (9)$$

An efficient way to compute the inverse of $\mathbf{C_{yy}}(t)$ is to apply the matrix inversion lemma. Since this process is totally same with the well-known RLS algorithm, the PAST just uses it for updating $\mathbf{W}(t)$ without further derivation. The PAST method requires $3nr + \boldsymbol{O}(r^2)$ computations at each update [3].

Note that the PAST method is derived by minimizing the modified cost function in $J'(\mathbf{W}(t))$ in (3). Therefore, the columns of estimated $\mathbf{W}(t)$ have lack of the orthonormality which depends on the signal-to-noise ratio (SNR) and forgetting factor β [3].

TABLE I
FAPI METHOD [7]

Initialization : $\mathbf{W}(0) = \begin{bmatrix} \mathbf{I}_r \\ \mathbf{0}_{(n-r)\times r} \end{bmatrix}$, $\mathbf{P}(0) = \mathbf{I}_r$
For $t = 1, 2, \cdots$
$\mathbf{y}(t) = \mathbf{W}^H(t-1)\mathbf{x}(t)$
$\mathbf{h}(t) = \mathbf{P}(t-1)\mathbf{x}(t)$
$\mathbf{g}(t) = \mathbf{h}(t) / [\beta + \mathbf{y}^H(t)\mathbf{h}(t)]$
$\varepsilon(t) = (\mathbf{x}^H(t)\mathbf{x}(t) - \mathbf{y}^H(t)\mathbf{y}(t))^{1/2}$
$\rho(t) = \mathbf{I}_p + \varepsilon^H(t)\left\{\mathbf{g}^H(t)\mathbf{g}(t)\right\}\varepsilon(t)$
$\tau(t) = \varepsilon^H(t)\left(\rho(t) + \rho^{\frac{1}{2}H}(t)\right)^{-1}\varepsilon(t)$
$\eta(t) = \mathbf{I}_p - \left(\mathbf{g}^H(t)\mathbf{g}(t)\right)\tau(t)$
$\mathbf{y}'(t) = \mathbf{y}(t)\eta(t) + \mathbf{g}(t)\tau(t)$
$\mathbf{h}'(t) = \mathbf{P}^H(t-1)\mathbf{y}'(t)$
$\boldsymbol{\varepsilon}(t) = \left\{\mathbf{P}(t-1)\mathbf{g}(t) - \mathbf{g}(t)\left[\mathbf{h}'^H(t)\mathbf{g}(t)\right]\right\}\left(\tau(t)\eta^{-1}(t)\right)^H$
$\mathbf{P}(t) = \frac{1}{\beta}\left\{\mathbf{P}(t-1) - \mathbf{g}(t)\mathbf{h}'^H(t) + \boldsymbol{\varepsilon}(t)\mathbf{g}^H(t)\right\}$
$\mathbf{e}'(t) = \mathbf{x}(t) - \mathbf{W}^H(t-1)\mathbf{y}(t)$
$\mathbf{W}(t) = \mathbf{W}(t-1) + \mathbf{e}'(t)\mathbf{g}^H(t)$
End

B. Fast Approximated Power Iteration (FAPI)[7]

An orthonormal subspace basis is required for some subspace-based estimation algorithms such as MUSIC [9]. The FAPI method applies new projection approximation instead of conventional projection approximation of the PAST method for orthonormal subspace basis as

$$\mathbf{W}(t) \simeq \mathbf{W}(t-1)\Theta(t)\,, \quad (10)$$

where $\Theta(t)$ is a $r \times r$ orthonormal matrix.

A family of power iteration methods, including the PAST and NP3 method, is based on conventional projection approximation by $\mathbf{W}(t) \simeq \mathbf{W}(t-1)$. These methods have a constraint that $\mathbf{R}(t)$ must be positive definite matrix. Hence, these subspace trackers could not guarantee to converge if $\mathbf{R}(t)$ deviates from positive definite matrix constraint. On the other hand, the FAPI is not affected by this constraint because the FAPI relies on the less restrictive approximation as (10). The best approximation of $\mathbf{W}(t)$ is obtained by solving the following minimization problem

$$\arg \min_{\Theta(t)} \left\| \mathbf{W}(t) - \mathbf{W}(t)\Theta(t) \right\|_F^2 , \quad (11)$$

and the solution of (11) is obtained by

$$\Theta(t)\Theta^H(t) = \left[\mathbf{I}_r + \mathbf{g}(t)\left(\mathbf{e}^H(t)\mathbf{e}(t)\right)\mathbf{g}^H(t)\right]^{-1}, \qquad (12)$$

where $\mathbf{I}_r$ is a $r \times r$ identity matrix, $\mathbf{g}(t)$ is a gain vector, and $\mathbf{e}(t)$ is an error vector. The particular solution of (12) is calculated by using the matrix inversion and related specific derivations can be found in [7]. The total procedure of the FAPI is presented in Table 1 and the FAPI requires only $n(3r+2)$ + $\boldsymbol{O}(r^2)$ computation at each update [7].

III. Gradient-based Variable forgetting factor Fast subspace tracking (GVFF FAPI) algorithm

A. GVFF Recursive Least Squares

Subspace tracking methods with a constant forgetting factor (FF), such as PAST, NP3, and FAPI, are not suitable for tracking in continuously time-varying subspace environments because its convergence rate is slow when the FF is close to one, whereas the misadjustment is large when the FF is small. In order to achieve the satisfactory performance of a subspace tracker in continuously time-varying environments, this paper applies the gradient-based variable forgetting factor (GVFF) RLS algorithm to the FAPI method. The GVFF RLS algorithm controls the forgetting factor by using the gradient based method which is derived from an improved mean square error (MSE) analysis of the time variable error weighting RLS (TWRLS) algorithm [10],[11]. The TWRLS algorithm can be obtained from the minimization of the least squares cost function parameterized by desired signal $\mathbf{d}(i)$, weight vector $\mathbf{w}(t)$, and error weighting function $\beta_{t-i}(t)$ as follows

$$J(t) = \sum_{i=0}^{t} \beta_{t-i}(t)\left\|\mathbf{d}(i) - \mathbf{w}^H(t)\mathbf{x}(u)\right\|^2 . \qquad (13)$$

The MSE analysis with error weighting function yields the following gradient updating equation [8]

$$\frac{\partial \sigma_{\mathbf{e}}^2(t+1)}{\partial \beta} = \varsigma_t \frac{\partial \sigma_{\mathbf{e}}^2(t)}{\partial \beta} + \frac{\partial \varsigma_t}{\partial \beta}\sigma_{\mathbf{e}}^2(t) + \frac{\partial h_t}{\partial \beta}\sigma_\eta^2, \qquad (14)$$

where $\partial\sigma_{\mathbf{e}}^2(t)/\partial\beta$ means a gradient of MSE, ς_t and h_t are coefficients to unite the multinomial of derivation process, $\sigma_{\mathbf{e}}^2(t)$ and σ_η^2 are the estimated MSE and the estimated measurement noise variance, respectively. Related specific derivations and parameters can be found in [8]. Using the gradient $\partial\sigma_{\mathbf{e}}^2(t)/\partial\beta$, the gradient based control for the variable forgetting factor of the TWRLS algorithm is expressed as

$$\beta(t) = \left[\beta(t-1) - \frac{\mu}{1-\beta(t-1)}\frac{\partial \sigma_e^2(t)}{\partial \beta}\right]_{2\beta_\#}^{\beta_{\max}} . \qquad (15)$$

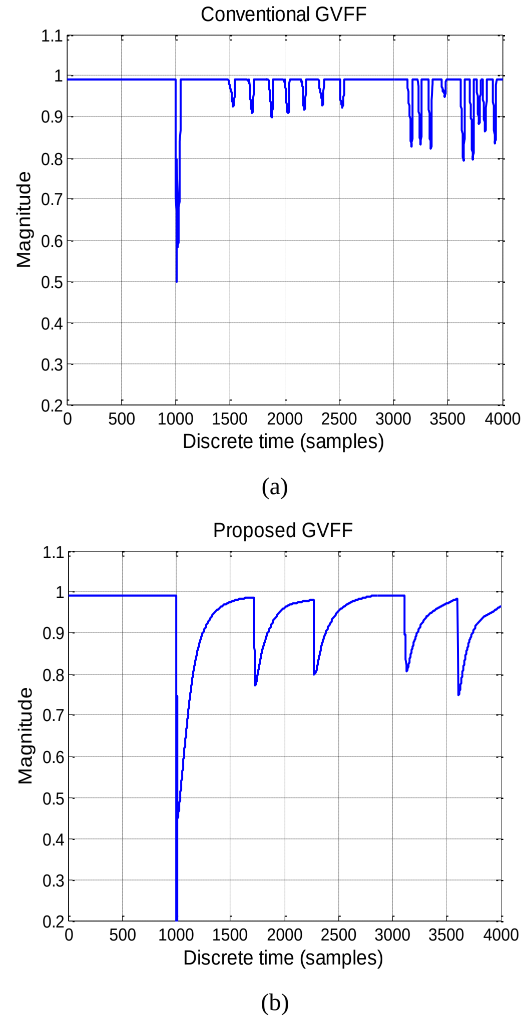


Fig. 1 The variable FF of GVFF FAPI : (a) using control equation (15), (b) using control equation (16)

The control mechanism can be explained as follows. If the derivative $\partial\varsigma_t/\partial\beta$ in (14) is positive, this enables the control equation to decrease the forgetting factor whenever there is a large increase in $\sigma_{\mathbf{e}}^2(t)$. On the other hand, $\partial h_t/\partial\beta$ in (14) is always negative and this drives the gradient to negative after $\sigma_{\mathbf{e}}^2(t)$ is reduced to a certain level and forces the forgetting factor toward the ceiling.

B. GVFF FAPI

The GVFF RLS algorithm is applied in FAPI method to improve the tracking performance in continuously time-varying environments. However, it is designed to yield better convergence performance in suddenly time-varying environments. This paper modifies the FF control equation in order to enhance the convergence performance in continuously time-varying environment as follows:

$$\beta(t)=\begin{cases}\left[\beta(t-1)-\dfrac{\mu}{\left(1-\beta(t-1)\right)^{2}}\dfrac{\partial\sigma_e^2(t)}{\partial\beta}\right]_{2\beta_\#}^{\beta_{\max}} & \textbf{for } \dfrac{\partial\sigma_e^2(t)}{\partial\beta}>0 \\ \left[\beta(t-1)-\dfrac{\mu}{\left(1-\beta(t-1)\right)^{-1/2}}\dfrac{\partial\sigma_e^2(t)}{\partial\beta}\right]_{2\beta_\#}^{\beta_{\max}} & \textbf{for } \dfrac{\partial\sigma_e^2(t)}{\partial\beta}\le 0\end{cases} \qquad (16)$$

The control mechanism can be explained as follows. The variable step-size $\mu/(1-\beta(t-1))^2$ in (16) enables the control to rapidly reduce the forgetting factor to a sufficiently small level whenever there is a small and positive $\partial\sigma_e^2(t)/\partial\beta$. On the other hand, the variable step-size $\mu/(1-\beta(t-1))^{-1/2}$ in (16) drives the forgetting factor to slowly increase when $\partial\sigma_e^2(t)/\partial\beta$ is negative. Figure 1 shows two cases of variable FF of the GVFF FAPI method that the first, controlled by the FF control equation (15), is represented in Fig. 1 (a) and another, controlled by the FF equation (16), is given in Fig. 1 (b).

IV. Performance Evaluation

To demonstrate the applicability in direction-of-arrival (DOA) systems and the tracking performances of the proposed method, we compared subspace error [6] and root mean square error (RMSE) for tracked DOA of GVFF FAPI with that of the conventional FAPI using the constant forgetting factor (FF). A linear uniform array with $n=8$ sensors and a time-varying DOA according source signal described in Fig. 2 were used in this simulation. The distance between the adjacent sensor elements was a half of the wavelength and the frequency of incident signal was 10 kHz. The step-size μ of the control equation of the GVFF FAPI method is 0.0001. SNR was 20dB and a sequence of snapshot $\mathbf{x}(t)$ (t = 1,2,⋯) was generated according to the above signal model and used to track the signal subspace by FAPI and GVFF FAPI. After each subspace was updated, the MUSIC algorithm was applied to compute RMSE of tracked DOA from signal subspace estimate. In this simulation, the proposed method using the control equation (15) and the control equation (16) refer to GVFF FAPI 1 and GVFF FAPI 2, respectively.

It is clear, from Fig. 3 that the subspace error and RMSE of tracked DOA about GVFF FAPI 1 and GVFF FAPI 2 are smaller than FAPI's. In addition, we can confirm that GVFF FAPI 2 shows the superior performance to GVFF FAPI 1 in continuously time-varying environment. However, the performances between GVFF FAPI 1 and GVFF FAPI 2 was reversed in the converging situation after DOA suddenly changed at t = 1000.

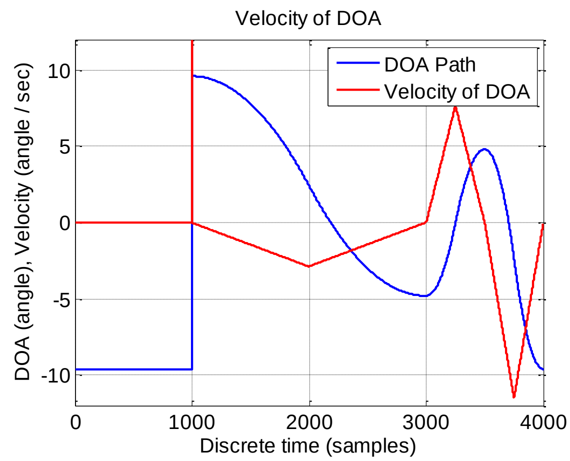
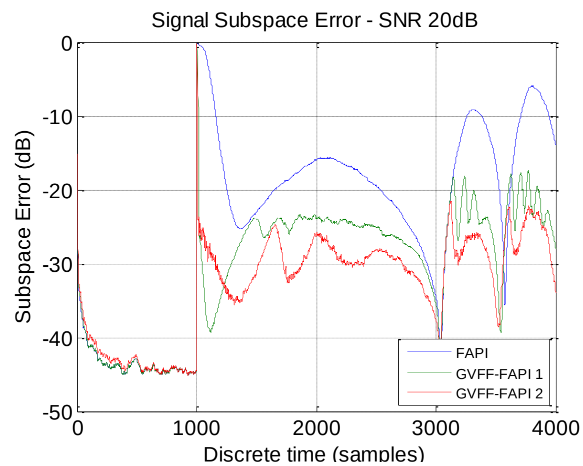
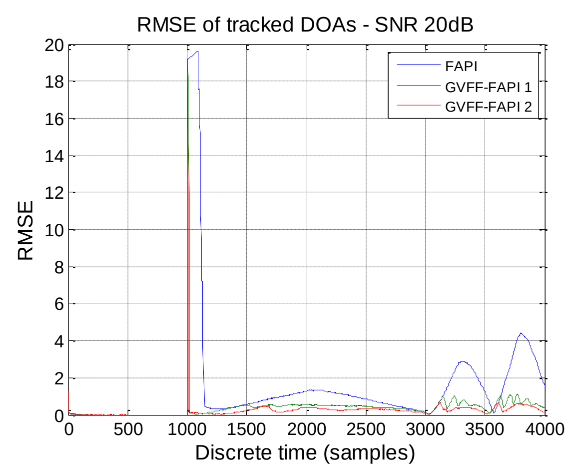


Fig. 2 Time varying DOA models of a source signal



(a)



(b)

Fig. 3 Performance comparison between FAPI and GVFF FAPI 1,2 : (a) subspace error, (b) RMSE of tracked DOA

V. Conclusion

In this paper, we proposed a subspace tracker named as the GVFF FAPI. In order to enhance the tracking performance of the FAPI in continuously time-varying environment, we modified the FF control equation of the GVFF RLS algorithm and applied the modified equation to the FAPI method. Simulation results show that the GVFF FAPI using the modified the FF control equation gives the outstanding performance than the FAPI and the GVFF FAPI using the original FF control equation.

The future works would be a theoretical analysis of the modified FF control equation and a comparative study withother's FF control equation of VFF RLS algorithms.

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